The Asymmetric Superfluid Local Density Approximation (ASLDA)

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Here we describe the form of the Asymmetric Superfluid Local Density Approximation (ASLDA), a Density Functional Theory (DFT) used to model the two-component unitary Fermi gas. We give the rational behind the functional, and describe explicitly how we determine the form of the DFT from the to the available numerical and experimental data.

I. INTRODUCTION

Here we describe the formulation of the Asymmetric Superfluid Local Density Approximation (ASLDA), which is a Density Functional Theory (DFT) describing normal and superfluid systems comprising two species of fermion. We show how the Monte-Carlo data of Refs. [1, 2, 3, 4, 5, 6, 7, 8] was incorporated into the ASLDA DFT used in the papers [9, 10]. In its present formulation, the ASLDA functional describes two species of fermions – denoted a (spin-up) and b (spin-down) – with equal masses $m_a = m_b = m$, and interacting through a resonant attractive inter-species s-wave two-body interaction described solely by the infinite two-body scattering length $a_s = \infty$. Throughout this paper, we use units where $\hbar = m = 1$ to simplify the notation.

The model is tuned to reproduce the thermodynamics of the homogeneous normal phase, and the homogeneous symmetric ($n_a = n_b$) superfluid phase (SF). Properties of these phases have been evaluated using accurate non-perturbative Monte-Carlo calculations, including the energy of the interacting normal state with varying degrees of polarization $n_a \neq n_b$ (Fig. 1), the energy of the SF phase $\xi = \mathcal{E}/\mathcal{E}_{FG} = 0.40(1)$ [2, 3], the quasi-particle dispersion relationship in the SF phase (Fig. 2) [2], and the effective mass of a spin-down fermion immersed in

a sea of spin-up fermions [4, 5, 6]. We describe here how to incorporate this high-quality information into the ASLDA functional, allowing it to accurately describe all the homogeneous normal and superfluid properties of the unitary Fermi gas. The ASLDA functional thus provides means of using the quantitative non-perturbative information about homogeneous phases to explore the structure of inhomogeneous systems. This has led to the prediction of a non-trivial supersolid Larkin-Ovchinikov (LO) phase [10] of the polarized unitary Fermi gas.

II. THE FUNCTIONAL

We choose to use an ASLDA functional with the following form,

$$
E = \int \mathrm{d}^3 \boldsymbol{r} \left\{ \mathcal{E}\big[n_{a,b}(\boldsymbol{r}), \tau_{a,b}(\boldsymbol{r}), \nu(\boldsymbol{r})\big] + \right.
$$
$$
\left. + V_{\mathrm{ext}}(\boldsymbol{r})\big[n_a(\boldsymbol{r}) + n_b(\boldsymbol{r})\big] + \mathrm{sources}\ldots \right\}, \quad (1)
$$

where the (local) energy-density $\mathcal{E}$ is a function the densities. We construct the densities from the two-component Bogoliubov (Nambu-Gor'kov) quasiparticle
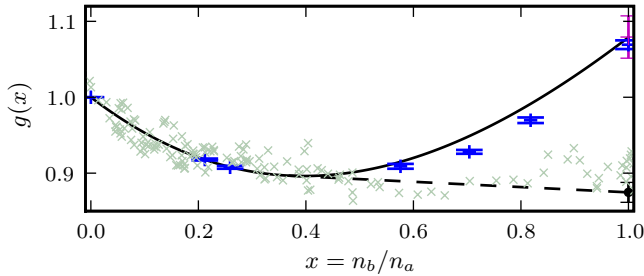

FIG. 1: (Color online) The dimensionless convex function $g(x)$ [11] that defines the energy density $\mathcal{E}(n_a, n_b) = \frac{3}{5} \frac{\hbar^2}{2m} (6\pi^2)^{2/3} [n_a g(x)]^{5/3}$. The points with error-bars (blue online) are the Monte-Carlo data from Refs. [4, 5, 6]. The fully-paired solution $g(1) = (2\xi)^{3/5}$ is indicated to the bottom right, and the recent MIT data [12] is shown (light $\times$) for comparison. The phase separation discussed in Refs. [4, 5, 6] is shown by the Maxwell construction (thin black dashed line).
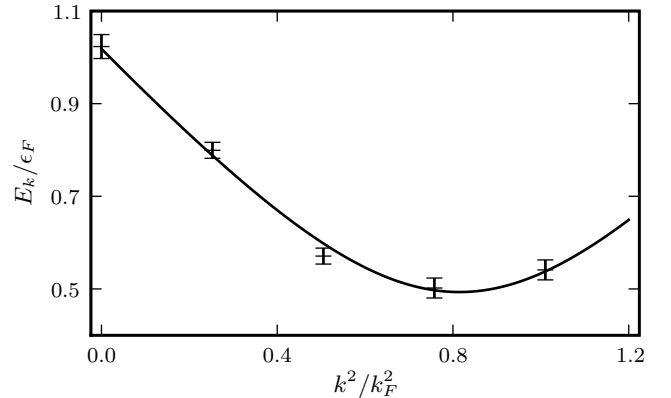

FIG. 2: Fit of the Monte-Carlo data for the quasiparticle dispersions from [2] with the BCS form (5). This, along with ξ is used to determine the parameters of the fully paired superfluid phase SF.

wavefunctions that satisfy the generalized Kohn-Sham equation $H_{KS} \cdot \psi_n = E_n \psi_n$ (see below):

$$\psi_n(\boldsymbol{r}) = \begin{pmatrix} u_n(\boldsymbol{r}) \\ v_n(\boldsymbol{r}) \end{pmatrix}. \tag{2}$$

The densities are constructed from these by summing over all eigenstates E_n weighted with the Fermi distribution function $f_\beta(E)$, where $\beta = 1/T$. (A small finite temperature can help the initial part of the self-consistent calculations to converge, see [13].)

$$n_a(\boldsymbol{r}) = \sum_n |u_n(\boldsymbol{r})|^2 f_\beta(E_n), \tag{3a}$$

$$n_b(\boldsymbol{r}) = \sum_n |v_n(\boldsymbol{r})|^2 f_\beta(-E_n), \tag{3b}$$

$$\tau_a(\boldsymbol{r}) = \sum_n |\boldsymbol{\nabla} u_n(\boldsymbol{r})|^2 f_\beta(E_n), \tag{3c}$$

$$\tau_b(\boldsymbol{r}) = \sum_n |\boldsymbol{\nabla} v_n(\boldsymbol{r})|^2 f_\beta(-E_n), \tag{3d}$$

$$\nu(\boldsymbol{r}) = \sum_n u_n(\boldsymbol{r}) v_n^*(\boldsymbol{r}) \frac{f_\beta(-E_n) - f_\beta(E_n)}{2}. \tag{3e}$$

We express our functional in terms of these densities (from here on we drop the explicit $\boldsymbol{r}$ dependence to simplify the notation)

$$\mathcal{E}\left[n_{a,b}, \tau_{a,b}, \nu\right] =$$
$$= \alpha_a(n_a, n_b) \frac{\tau_a}{2} + \alpha_b(n_a, n_b) \frac{\tau_b}{2} +$$
$$D(n_a, n_b) + g_{\text{eff}}(n_a, n_b) \nu^* \nu \tag{4a}$$

where $\alpha_{a,b}(\lambda n_a, \lambda n_b) = \alpha(n_a, n_b)$ are the inverse effective masses, $D(\lambda n_a, \lambda n_b) = \lambda^{5/3} D(n_a, n_b)$ contains the density dependent portion of the functional and $C(\lambda n_a, \lambda n_b) = \lambda^{1/3} C(n_a, n_b)$ is an appropriate density dependent inverse effective coupling, with regulator Λ:

$$\frac{1}{g_{\text{eff}}} = C(n_a, n_b) - \Lambda. \tag{4b}$$

The divergences appear in both the pairing and kinetic terms such that $\Delta = -g_{\text{eff}} \nu$ and the combination

$$\alpha_a \frac{\tau_a}{2} + \alpha_b \frac{\tau_b}{2} - g_{\text{eff}} \nu^* \nu \tag{4c}$$

are finite. Thus, the energy functional may be expressed in terms of finite combinations

$$\mathcal{E} = \left[\alpha_a \frac{\tau_a}{2} + \alpha_b \frac{\tau_b}{2} - \Delta^* \nu\right] + D \tag{4d}$$

where $\Delta^* \nu = \nu^* \Delta = -g_{\text{eff}} \nu^* \nu$. One recovers a typical Kohn-Sham functional by setting $\alpha_{a,b} \equiv 1$ and $\Delta \equiv 0$ ($\nu \equiv 0$). Regularization is required to describe superfluids ($\Delta \neq 0$ and $\nu \neq 0$) because the anomalous density ν and the kinetic energy densities $\tau_{a,b}$ diverge. This regulator dependence is removed when one considers the

proper combinations of terms described above. Allowing the effective mass to deviate $\alpha \neq 1$ in the SLDA description of the trapped fermions improved the agreement with Monte-Carlo (MC) results [9, 14, 15, 16], and can be constrained by non-perturbative results, so we keep this generalization. The coefficient functions $\alpha_a(n_a, n_b)$, $\alpha_b(n_a, n_b)$, $C(n_a, n_b)$, and $D(n_a, n_b)$ are chosen to match the thermodynamic properties of the system in the homogeneous limit. The lack of scales in the unitary limit further constrains the forms of these coefficients to be homogeneous functions of specific degree in the densities $n_{a,b}$.

We established this form of the ASLDA functional using the following guiding principles:

Gradient Expansion: We start by formulating a functional applicable to slowly varying systems. To this end, we have neglected gradient terms. Indeed, the symmetric SLDA functional has had remarkable success *without any* gradient corrections [9], so we assume that this property holds for asymmetric systems and start with only local terms. An important future direction will be to quantify the effects of gradient corrections, and to extract the coefficients of the leading gradient terms.

Note that the success of the SLDA implies that the subset of gradient corrections included implicitly through the standard kinetic terms provides the dominant gradient contribution. These corrections are also included in the ASLDA.

To justify the omission of gradient terms (beyond those contained in $\tau_{a,b}$), consider the lowest order gradient correction to the SLDA functional: this is of the form $|\boldsymbol{\nabla} n|^2/n$ and would lead to a correction in the total energy of a system of N harmonically trapped fermions that scales as $N^{2/3}$. The extremely good agreement between the SLDA functional in Ref. [9] and the *ab initio* results of Refs. [14, 15, 16] indicate that the coefficient of this correction must be extremely small. One can also show that, in the dilute limit, the strength of the gradient terms is controlled by the size of the length scales such as the effective range and p-wave scattering length of the interaction [17, 18]. In the unitary Fermi gas, both of these vanish.

In addition, the leading gradient corrections that could appear away from the symmetric limit $x = n_b/n_a = 1$ are suppressed by $(1 - x)^2$. Thus, we expect them to have little effect on phases close to the fully paired SF state, for example, the LO phase discussed in [10].

Finally, Galilean invariance requires that if $\alpha_{a,b} \neq 1$ then $\tau_{a,b}$ be replaced by $\tau_{a,b} - \boldsymbol{p}_{a,b}^2/n_{a,b}$ where $\boldsymbol{p}_{a,b}$ is the local current density [9]. This correction is only required to discuss states that break time-reversal invariance (i.e. that contain currents).

Simplicity: Due to the fermion sign problem, there are

very few reliable calculations of properties in the polarized regime, thus there is not enough data to properly constrain a fully general functional. Monte-Carlo simulations, however, have provided reasonably constraints on the form of the normal state energy density, thus we allow for full generality in terms of the functional dependence on the densities, but restrict the general dependence on the anomalous and kinetic densities. The most general form of local function would allow the unknowns to depend on all of the dimensionless combinations of densities, including for example, an arbitrary dependence on the dimensionless regulator-invariant combination $[\alpha_a \tau_a/2 + \alpha_b \tau_b/2 - \Delta^* \nu]/n^{5/3}$. Presently, we see no need for this added complication.

Quasiparticle Dispersion: Monte-Carlo calculations about the symmetric phase have suggested that the low-temperature quasiparticle dispersions are well described by the BCS form

$$E_k = \sqrt{\left[\alpha \frac{k^2}{2} + (U - \mu)\right]^2 + |\Delta|^2} \qquad (5)$$

where α, $(U - \mu)$, and Δ are effective parameters. This form of dispersion follows from introducing the anomalous pairing density ν through the quadratic form $\nu^* \nu$ along with canonical kinetic terms.

Decoupling of Paired and Normal States: In our actual formulation, we also neglect the general density dependence of the pairing interaction $C(n_a, n_b)$, replacing this by the same term with a single constant used in the SLDA. The justification of this is two-fold: 1) The SLDA worked very well, even including small polarizations. 2) The energy of interaction of the normal state is well approximated by considering only the parameters of the SLDA without any additional density dependence.

Together these suggest that the physics of the superfluid state somewhat decouples from the physics of the normal state, allowing one to characterize the fully paired superfluid independently of the normal state.

This implies a qualitative ansatz of this functional: that the structure of the polarized phases arises from the competition between the fully paired superfluid physics and the interacting normal state physics. In principle, it is possible that some qualitatively new description is required to properly account for the structure of polarized fermionic matter – for example to describe the appearance of p-wave pairing at large polarizations [19] – but to fix such a description will require high precision *ab initio* calculations and/or experiments that have not yet been realized.

Varying this functional with respect to the quasi-particle wavefunctions and occupation numbers gives the generalized Kohn-Sham equation $H_{KS} \cdot \psi_n = E_n \psi_n$, where

$$H_{KS} = \begin{pmatrix} K_a - \mu_a + U_a & \Delta^* \\ \Delta & -K_b + \mu_b - U_b \end{pmatrix}, \qquad (6a)$$

and the kinetic and potential operators are

$$K_a u = -\frac{1}{2} \boldsymbol{\nabla} \cdot (\alpha_a \boldsymbol{\nabla} u) \qquad (6b)$$

$$U_a = \frac{\partial \alpha_-}{\partial n_a} \frac{\tau_-}{2} + \frac{\partial D}{\partial n_a} - \frac{\partial C}{\partial n_a} |\Delta|^2 + $$
$$+ \frac{\partial \ln \alpha_+}{\partial n_a} \left[\left(\alpha_+ \frac{\tau_+}{2} - \Delta^* \nu\right) - C|\Delta|^2 \right], \qquad (6c)$$

with the notations

$$\alpha_\pm = \frac{\alpha_a \pm \alpha_b}{2}, \quad \tau_\pm = \tau_a \pm \tau_b, \quad n_\pm = n_a \pm n_b. \qquad (6d)$$

The form of these operators for species b are obtained by interchanging $a \leftrightarrow b$. Note that the terms have been grouped so that the ultraviolet (UV) divergences arising from the local form of the anomalous density ν cancel in the last term (see Refs. [9, 20] for details). All other terms are finite.

III.   FITTING THE FUNCTIONAL

To further specify the functional, we must fix the forms of the functions $\alpha_{a,b}(n_a, n_b)$, $C(n_a, n_b)$, and $D(n_a, n_b)$. To do this, we characterize the thermodynamic properties of the system, which are fortunately quite tightly constrained [11], and have both calculational [4, 5, 11, 21] and experimental [12, 22] verification.

The form of the functional allows us to consider two different species (for example, with different masses), but we are interested in systems with two identical species. The functional must thus exhibit the discrete symmetry $a \leftrightarrow b$. This constrains the form of the functions $\alpha_{a,b}$, D and C:

$$\alpha_a(n_a, n_b) = \alpha_b(n_b, n_a) \qquad (7a)$$

$$D(n_a, n_b) = D(n_b, n_a) \qquad (7b)$$

$$C(n_a, n_b) = C(n_b, n_a). \qquad (7c)$$

Dimensional analysis determines the overall scaling of the functions, and we may fully parametrize the functional with three dimensionless functions $\alpha(x)$, $b(x)$, and $\gamma(x)$ of the asymmetry $x = n_b/n_a \in [0, 1]$, with the complementary region determined by symmetries (7):

$$\alpha_a(n_a, n_b) = \alpha(x), \qquad (8a)$$

$$\alpha_b(n_a, n_b) = \alpha(1/x), \qquad (8b)$$

$$D(n_a, n_b) = \frac{(3\pi^2)^{5/3}(n_a + n_b)^{5/3}}{10\pi^2} \beta(x), \qquad (8c)$$

$$C(n_a, n_b) = \frac{(n_a + n_b)^{1/3}}{\gamma(x)}. \qquad (8d)$$

As a technical note, we ensure that our parametrization is smooth at $x = 1$ by letting $\alpha(x)$ be smooth over $[0, \infty)$, and by forming smooth even functions over the variable $z \in [-1, 1]$ where $\ln x = c \tanh^{-1} z$: thus $\tilde{\beta}(z) = \beta(x(z))$ and $\tilde{\gamma}(z) = \gamma(x(z))$ are smooth at $x = 1$ if and only if $\tilde{\beta}(z)$ and $\tilde{\gamma}(z)$ are smooth even functions. The choice of the parameter c is made so that the interpolations of the Monte-Carlo data are well behaved.

To determine these dimensionless functions, we match the functional to the Monte-Carlo calculations of pure and homogeneous thermodynamic phases. These phases possess no gradients, and so solving the density functional for homogeneous matter is equivalent to performing a simple Thomas Fermi type calculation with the added complication that the parameters: $m_{a,b}$, etc. depend on the densities, which must be determined self-consistently. This gives rise to a set of non-linear set of equations that can be fairly easily solved.

We start with the homogeneous fully paired superfluid phase. This is described by the three numbers $\alpha = \alpha(1)$, $\beta = \beta(1)$, and $\gamma = \gamma(1)$ which may be extracted by fitting the quasi-particle dispersion relationship and the energy. As discussed earlier, we simplify the functional dependence of the function $\gamma(x) = \gamma$ by simply keeping this constant. The remaining functional forms $\alpha(x)$ and $\beta(x)$ are determined by fitting the energy of the homogeneous normal state to Monte-Carlo results. This completely specifies the functional in a unique manner as we shall now describe.

A. Symmetric Superfluid Properties:

As suggested in [9], by considering the calculated properties of the fully paired symmetric superfluid, one may determine the values of the functions $\alpha(1)$, $\beta(1)$, and $\gamma(1)$ at the point $x = n_b/n_a = 1$. We start by taking

$$n_a = n_b = n = \frac{n_+}{2} \tag{9}$$

where $n_+ = n_a + n_b = 2n$ is the total density. Dimensional analysis determines the following forms of the derivatives (evaluated at $n_a = n_b = n$):

$$\frac{\partial \alpha_+}{\partial n_a} = \frac{\partial \alpha_+}{\partial n_b} = 0, \tag{10a}$$

$$\frac{\partial C}{\partial n_a} = \frac{\partial C}{\partial n_b} = \frac{C}{3n}, \tag{10b}$$

$$\frac{\partial D}{\partial n_a} = \frac{\partial D}{\partial n_b} = \frac{5D}{3n}. \tag{10c}$$

Thus, the effective potentials $U_a = U_b = U$ have the following simplified form,

$$U(n, n) = \frac{5D(n, n)}{3n} - \frac{C(n, n)}{3n} \Delta^* \Delta, \tag{11}$$

and one may take a linear combination of the kinetic terms to obtain

$$K_+ = \frac{K_a + K_b}{2} = -\frac{\alpha_+ \boldsymbol{\nabla}^2}{2}. \tag{12}$$

Thus, the symmetric phase depends only on three parameters $\alpha = \alpha(1)$, $\beta = \beta(1)$, and $\gamma = \gamma(1)$ via:

$$\alpha_+(n, n) = \alpha(1), \tag{13a}$$

$$C(n, n) = \frac{n_+^{1/3}}{\gamma(1)}, \tag{13b}$$

$$D(n, n) = \beta(1) \frac{(3\pi^2 n_+)^{5/3}}{10\pi^2} = \beta(1)\, \mathcal{E}_{FG}. \tag{13c}$$

For a given inverse effective mass α, the other two parameters γ and β may be fit by requiring that the energy and spectral gap satisfy

$$\mathcal{E}_{SF} = \mathcal{E}(n, n) = \xi \mathcal{E}_{FG} = \xi \frac{(3\pi^2 n_+)^{5/3}}{10\pi^2}, \tag{14a}$$

$$\Delta = \eta \epsilon_F = \eta \frac{(3\pi^2 n_+)^{2/3}}{2}, \tag{14b}$$

where

$$k_F = (3\pi^2 n_+)^{1/3}, \tag{15a}$$

$$\epsilon_F = \frac{k_F^2}{2} = \frac{(3\pi^2 n_+)^{2/3}}{2}, \tag{15b}$$

$$\mathcal{E}_{FG} = 2\frac{k_F^5}{20\pi^2} = \frac{(3\pi^2 n_+)^{5/3}}{10\pi^2} = \frac{3}{5} n_+ \epsilon_F. \tag{15c}$$

The $T = 0$ symmetric state is characterized by the integrals

$$n_+ = \int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3} \left[1 - \frac{\epsilon_k}{E_k} \right], \tag{16a}$$

$$\mathcal{E}_{SF} = \int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^2} \left\{ \alpha \frac{k^2}{2} \left[1 - \frac{\epsilon_k}{E_k} \right] - \frac{|\Delta|^2}{2E_k} \right\} + \beta \mathcal{E}_{FG}, \tag{16b}$$

$$C = \frac{n_+^{1/3}}{\gamma} = -\int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3} \left[\frac{1}{2E_k} - \frac{1}{\alpha k^2} \right], \tag{16c}$$

where

$$\epsilon_k = \alpha \frac{k^2}{2} + (U - \mu), \tag{16d}$$

$$E_k = \sqrt{\epsilon_k^2 + |\Delta|^2}. \tag{16e}$$

Given fixed values of α, ξ, and η, we proceed as follows: 1) Choosing the density n so that $\epsilon_F = 1$: This fixes the scale and determines Δ, 2) Use equation (16a) to solve for the combination $U - \mu$ that appears on the right hand side through ϵ_k/E_k, 3) Use equation (16b) and ξ to solve for β, and 4) Use equation (16c) to solve for γ.

The parameters ξ and $\eta = \Delta/\epsilon_F$ have been measured by several Monte-Carlo techniques [1, 2, 3, 7, 8]. We take

the following values in our estimates [2, 3]:

$$\xi = \frac{\mathcal{E}(n,n)}{\mathcal{E}_{FG}(n,n)} = 0.40(1), \quad \eta = \frac{\Delta}{\epsilon_F} = 0.504(24). \quad (17)$$

Note that it was incorrectly stated in [9] that $\alpha_{x=1}$ could also be determined through the condition that the chemical potential satisfy $\mu = \xi\epsilon_F$, but a careful examination shows that this is implied by (14). It is also clear in DFT's developed perturbatively [17, 18] that the effective mass is arbitrary. In order to fix the effective mass, we match the quasi-particle dispersion relationship as determined from the Monte-Carlo results [2]. The dispersion relationships within our density functional have the form

$$\frac{E_k}{\epsilon_F} = \sqrt{\left[\frac{\alpha}{2}\frac{k^2}{k_F^2} + \left(\frac{U}{\epsilon_F} - \xi\right)\right]^2 + \left(\frac{\Delta}{\epsilon_F}\right)^2}. \quad (18)$$

Note that the combination $U/\epsilon_F - \xi$ is fixed from (16a) and depends only on Δ/ϵ_F, so the quasiparticle dispersion relation is sensitive only to Δ/ϵ_F and the effective mass. We add the value (17) as an additional data-point in the fit and perform a non-linear least-squares fit.

The fit to the Carlson-Reddy is shown in Fig. 2 and gives the following parameter values:[1]

$$\alpha_{x=1} = m_{\text{eff}}^{-1}/m^{-1} = 1.094(17), \quad (19a)$$
$$\beta_{x=1} = -0.526(18), \quad (19b)$$
$$\gamma^{-1} = -0.0907(77) \quad (19c)$$
$$\bar{\beta} = U/\epsilon_F = -0.491(18), \quad (19d)$$
$$\eta = \Delta/\epsilon_F = 0.493(12), \quad (19e)$$
$$\xi_N = \alpha + \beta = 0.567(24). \quad (19f)$$

where

$$\frac{U}{\epsilon_F} = \beta - \frac{(3\pi^2)^{2/3}}{6\gamma}\left(\frac{\Delta}{\epsilon_F}\right)^2. \quad (20)$$

In principle, one should use some form of *ab initio* calculation or experimental measurement for polarized systems to determine the dependence of the parameters α, β, and γ on the polarization $x = n_b/n_a$. Unfortunately, the fermion sign problem has made this difficult and there is presently insufficient quality data to perform such a fit. Instead, we make the approximation that

$$\gamma(x) = \gamma(1) = \text{const.} \quad (21)$$

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[1] Here we have performed a simple three-parameter non-linear least-squares fit: this has a quality factor $Q = 0.52$ which is quite good. A different analysis would hold Δ and U fixed to the properly determined values, but the method here is consistent since the errors are of the same magnitude.

B. Normal State

The remaining functional forms for $\alpha(x)$ and $\beta(x)$ can be extracted from properties of the homogeneous normal state at $T = 0$ by explicitly setting the anomalous density $\nu = 0$. This metastable state may be explored by choosing a suitable nodal approximation in the fixed-node Monte-Carlo (FN-MC) calculations. Such a restriction seems to remove most of the superfluid correlations from the results, but we suspect that one cannot completely remove all contamination from the superfluid state for small polarizations. For this reason, we have only included the data for large polarizations $x = n_b/n_a < 0.5$ in our fitting of the parameters for the functional (see Fig. 1 and Tab. II).

The energy-density for the normal phase of homogeneous matter has the form

$$\mathcal{E}[n_a, n_b] = \frac{\alpha_a(6\pi^2 n_a)^{5/3}}{20\pi^2} + \frac{\alpha_b(6\pi^2 n_b)^{5/3}}{20\pi^2} + D$$
$$= \frac{(6\pi^2)^{5/3}(n_a + n_b)^{5/3}}{20\pi^2}G(x) \quad (22)$$

where

$$G(x) = \frac{\alpha(x)}{(1+x)^{5/3}} + \frac{\alpha(x^{-1})}{(1+x^{-1})^{5/3}} + 2^{-2/3}\beta(x),$$
$$= \frac{1}{(1+x)^{5/3}}g^{5/3}(x). \quad (23)$$

From this relationship, one can uniquely determine the functional form for $\beta(x)$ given a form for the inverse effective mass $\alpha(x)$, which we shall construct below, and the normal state energy density $g(x)$, which has been well-constrained by Monte-Carlo data [4] (see Fig. 1).

C. Effective Mass Parametrization: $\alpha(x)$

Several different Monte-Carlo calculations have constrained the effective "polaron" mass of a single spin-down fermion in a sea of spin-up fermions, $m_0 \approx$

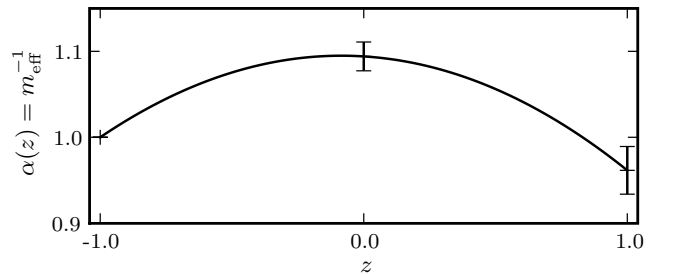
FIG. 3: Inverse effective mass $\alpha = m_{\text{eff}}^{-1}$ as a function of $z = \tanh\frac{1}{c}\ln x$ with $c = 1$. The function is a parabolic fit through the three data-points.

1.04(3)m [4, 5, 6], and the effective mass of the particles in the background gas will not be altered in the limit of extreme polarization. Thus, the endpoints of the function $\alpha(x)$ are constrained: $\alpha(0) = 1$, $\alpha(\infty) = 0.96(3)$. A third point is obtained from the effective mass $\alpha(1) = 1.09(2)$ in the fully paired superfluid phase (19e). We use these to provide a smooth interpolation via the parametrization

$$z_c = \tanh \frac{\ln x}{c} \qquad (24)$$

as shown in Fig. 3.

We use the variable z_c (24) here so that the interpolation is over a finite range $z_c \in [-1, 1]$. The parameter c gives us some control over the shape of the resulting curve as demonstrated in Fig. 4. Note that for $c < 2$, $\mathrm{d}z_c/\mathrm{d}x|_{x=0} = 0$, hence the function $\alpha(x)$ will be flat at $x = 0$. The addition of a few spin-down particles should not affect the mass of the spin-up particle: most of the change should occur when one approaches equal densities. There is not really sufficient information to further characterize this parametrization, but the effective mass does not vary much, so we do not expect this to be a significant source of error.

Including density dependent inverse masses $\alpha_{a,b}(n_a, n_b)$ can be of quantitative importance, but does not significantly alter the qualitative aspects of the ASLDA (such as presented in [10]). Thus, to obtain qualitative results, it can be a good first approximation to simply use the functional with $\alpha_{a,b}(n_a, n_b) = 1$.

D. "Hartree" Energy : $\beta(x)$.

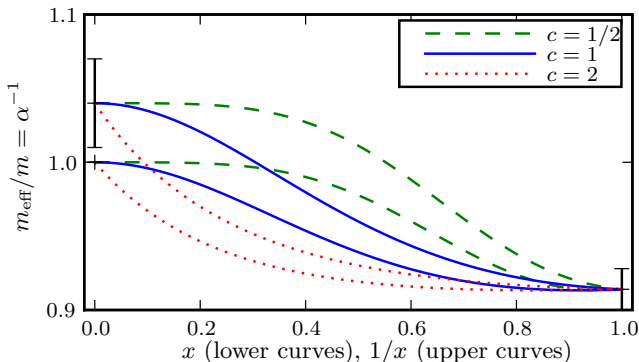To finish the parametrization, we must provide an interpolation of the function $g(x)$. We provide an interpo-

FIG. 4: (Color online) Effective mass as a function of $x = n_b/n_a$ for several values of c: $c = 1/2$ – dashed green, $c = 1$ – solid blue, $c = 2$ – dotted red. The upper curves represent the extension of the curves to $x > 1$ with abscissa $1/x$. Since the effective mass is not likely to change rapidly near $x = 0$ (especially for the majority component), the parameter c is probably best kept near 1 and we use $c = 1$.

lation for the function $G(x)$ (23) rather than directly for $\beta(x)$ or $g(x)$ because: 1) It is finite everywhere and, 2) such an interpolation is independent of the inverse effective mass function $\alpha(x)$. Again, we use the parametrization (24) with $z_c \in [-1, 1]$ with the assumption that $G(x)$ is smooth at $x = 1$, which requires that the interpolated function $\tilde{G}(z) = G(x(z))$ is smooth and even. This is quite easy to do (one can always just explicitly make the interpolation even $[\tilde{G}(z) + \tilde{G}(-z)]/2$). The assumption of smoothness also gives a non-trivial constraint on the data which should admit a smooth interpolation at $z_c = 0$. The resulting function is shown in Fig. 5 for several values of c. Here the value of $c = 2$ is best because $\mathrm{d}z_c/\mathrm{d}x|_{x=0} = 2$ is finite, preserving the structure of the interpolation. (The other values of c drastically affect the slope of the interpolated $g(x)$ without any physical motivation.) The preference for $c = 2$ is most evident in Fig. 6.
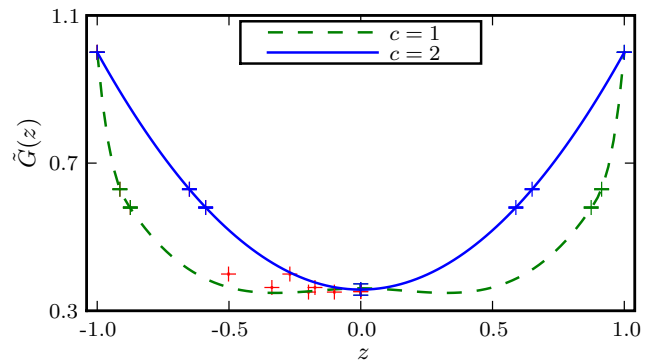

FIG. 5: (Color online) The function $\tilde{G}(z)$ defined in (23) plotted for various values of c used to define the interpolation through the coordinate z_c (24): $c = 1$ – dashed green, $c = 2$ – solid blue. See Fig. 6 which better emphasizes how this value is a better fit. We omitted the red points from our interpolation as discussed in the text.

IV. LOFF

Finally we include the derivation of a simple relationship between the average pressure and energy density of a LOFF state with period L imposed by boundary conditions. The physical solution will have a spontaneously determined length scale L, but our calculation must impose this through boundary conditions. To model these states, we use a periodic basis, summing over transverse and Bloch momenta. One must then adjust L to maximize the average pressure.

Consider the form of the average pressure at unitarity. The lack of scales implies

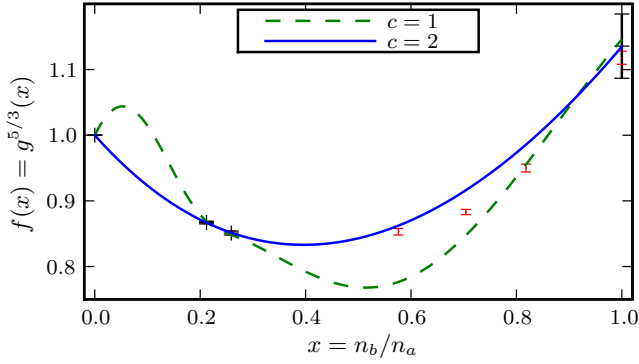$$s^5 \mathcal{P}(s^{-2}\mu_a, s^{-2}\mu_b, sL) = \mathcal{P}(\mu_a, \mu_b, L). \qquad (25)$$

FIG. 6: (Color online) The function $f(x) = g^{5/3}(x)$. Both function plotted for various values of c used to define the interpolation through the coordinate z_c (24): $c = 1$ – dashed green, $c = 2$ – solid blue. It can be clearly seen here that the value of $c = 2$ is preferred for a smooth interpolation.

TABLE I: Interpolation points for the function $\tilde{\alpha}(z)$.

z	$\tilde{\alpha}$
-1	1.000(00)
0	1.094(17)
1	0.962(28)

TABLE II: Interpolation points for the function $\tilde{G}(z)$.

z	$\tilde{G}$
-1.0000	1.0000(00)
-0.6502	0.6293(15)
-0.5886	0.5797(20)
0.0000	0.3577(15)
0.5886	0.5797(20)
0.6502	0.6293(15)
1.0000	1.0000(00)

Differentiating and using the thermodynamic relationship $n_{a,b} = \partial\mathcal{P}/\partial\mu_{a,b}$, we obtain:

$$5\mathcal{P} - 2\mu_a n_a - 2\mu_b n_b + L\frac{\partial\mathcal{P}}{\partial L} = 0. \qquad (26)$$

Coupled with the thermodynamic relationship $\mathcal{P} = \mu_a n_a + \mu_b n_b - \mathcal{E}$ we have

$$X = L\frac{\partial\mathcal{P}}{\partial L} = 2\mathcal{E} - 3\mathcal{P}. \qquad (27)$$

When the pressure is maximized, this quantity $X = 0$. Note that this relationship is derived solely on dimensional grounds.

V. INTERPOLATIONS:

The functional (4a) is completely described by the three functions $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ through equations (8). These functions are defined by the functions $\tilde{\alpha}(z)$ and $\tilde{G}(z)$ that interpolate the data in tables I and II. The resulting weighted cubic-spline interpolations are shown in Fig. 3 and Fig. 5) respectively.

$$\alpha(x) = \tilde{\alpha}\left(\tanh\ln x\right),$$
$$\beta(x) = 2^{2/3}\tilde{G}\left(\tanh\frac{\ln x}{2}\right) +$$
$$- \frac{\alpha(x)}{(1+x)^{5/3}} - \frac{\alpha(x^{-1})}{(1+x^{-1})^{5/3}},$$
$$\gamma(x) = \gamma.$$

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